On weak convergence of the empirical process with random sample size

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Introduction. Let $U_1, U_2, \ldots, U_n$ be a random sample taken from the uniform distribution on [0, 1] and let $F_n(t)$ be their empirical distribution function and $Y_n(t)=\sqrt{n}\,(F_n(t)-t)$ the empirical process. If $D=D_{[0,1]}$ denotes the space of functions on [0, 1] having discontinuities only of the first kind, endowed with the Skorohod topology (see [4]) and $\mathscr{D}$ denotes the σ-algebra generated by the open sets of this topology, then $Y_n(t)$ (defined on some probability space $\{\Omega, \mathscr{B}, P\}$) is a random function of $\{D, \mathscr{D}\}$. Here and throughout this paper we use the standard terminology and notation of BILLINGSLEY's book [4] (see also [10] in these *Acta*). As well known, [4] or [12], the empirical process Y_n weakly converges (as $n\to\infty$) to the Brownian Bridge W° with covariance function $s(1-t)$ for $0\leqq s\leqq t\leqq 1$:

$$Y_n \xrightarrow{\mathscr{D}} W^\circ. \tag{1}$$

Let us consider the following functionals of $Y_n(t)$:

$$\sup_{0\leqq t\leqq 1} |Y_n(t)| \text{ — Kolmogorov's statistic,} \tag{2}$$

$$\sup_{0\leqq t\leqq 1} Y_n(t) \text{ — Smirnov's statistic,} \tag{3}$$

$$\int_0^1 [Y_n(t)]^2 f(t)\,dt \text{ — Cramér—von Mises statistic,} \tag{4}$$

$$\sup_{\alpha\leqq t\leqq\beta} \frac{|Y_n(t)|}{g(t)} \tag{5}$$

Anderson and Darling—Rényi statistics,

$$\sup_{\alpha\leqq t\leqq\beta} \frac{Y_n(t)}{g(t)} \tag{6}$$

where $f(t)$ and $g(t)$ are some non-negative weight functions, and on the interval $[\alpha, \beta]$, $[\alpha, \beta]\subseteqq[0, 1]$, the function $g(t)$ is bounded away from zero. ANDERSON and DARLING [1] particularly dealt with $g(t)=\sqrt{t(1-t)}$ and RÉNYI [17] with $g(t)=t$ and $1-t$. Since all the above functionals are continuous in the Skorohod topology, we have, as a consequence of relation (1), that the distributions of the statistics (2)—(6) converge to the distributions of the appropriate functionals of W° (cf. [12] and [8]).

Let now, for each n, ν_n be a positive valued random variable defined on the same probability space $\{\Omega, \mathcal{B}, P\}$. In [16], PYKE explains the importance of dealing with the random sample size empirical process $Y_{\nu_n}(t)$, that is, when at each given time n, the size of the sample is the random ν_n. He proves that if the variables ν_n are such that ν_n/n converges in probability (denoted from now on by $\xrightarrow{P}$) to 1, then

(7) $$Y_{\nu_n} \xrightarrow{\mathcal{D}} W^\circ.$$

As to the behaviour of ν_n, in [9] we used the more general condition that $\nu_n/n \xrightarrow{P} \nu$, where ν is an arbitrary positive random variable. There, in [9], we constructed a partial sum type process $X_n(t)$, for which the distributions of

$$\sup_{0\le t\le 1} |X_{\nu_n}(t)| \quad \text{and} \quad \sup_{0\le t\le 1} X_{\nu_n}(t)$$

for large n are the same as those of

$$\sup_{0\le t\le 1} |Y_{\nu_n}(t)| \quad \text{and} \quad \sup_{0\le t\le 1} Y_{\nu_n}(t),$$

and proved that $X_{\nu_n} \xrightarrow{\mathcal{D}} W^\circ$. That is, we proved that the random sample size KOLMOGOROV—SMIRNOV statistics of (2) and (3) have the same limit distributions as those of the originals. The aim of the present paper is to prove directly relation (7) under this latter condition on ν_n, i.e. $\nu_n/n \xrightarrow{P} \nu$, which is also the most frequently used condition in the theory of limit distributions of sequences of random variables with random indices (cf. [13]) in general.

The Results. Theorem 1. *If Y_n denotes the empirical process and W° the Brownian Bridge, and if the sequence of positive integer valued random variables ν_n is such that*

$$\frac{\nu_n}{n} \xrightarrow{P} \nu,$$

where ν is a positive random variable, then

$$Y_{\nu_n} \xrightarrow{\mathcal{D}} W^\circ.$$

A considerable part of the literature dealing with the empirical process is devoted to finding representations (in distribution) of the empirical distribution function which would easier lend themselves to analysis. (See e.g. RÉNYI [17], BREIMAN [6], BRILLINGER [7], PYKE [16], MÜLLER [15]). One of these can be described as follows. Let $Z_k = \xi_1 + \ldots + \xi_k$ be the partial sum sequence of independent exponential random variables ξ_n with mean 1, and let $U_1^{(n)}, U_2^{(n)}, \ldots, U_n^{(n)}$ denote the order statistics of the sample $U_1, \ldots, U_n$ of the Introduction. Then the joint distribution of $U_1^{(n)}, U_2^{(n)}, \ldots, U_n^{(n)}$ is the same as that of $Z_1/Z_{n+1}, \ldots, Z_n/Z_{n+1}$ (see BREIMAN [6]). Consequently, if we define the random functions (of $\{D, \mathscr{D}\}$)

$$G_n(x) = \begin{cases} 0, & \dfrac{Z_1}{Z_{n+1}} > x, \\[2ex] \dfrac{k}{n}, & \dfrac{Z_k}{Z_{n+1}} \leqq x < \dfrac{Z_{k+1}}{Z_{n+1}}, \quad k = 1, \ldots, n-1, \\[2ex] 1, & \dfrac{Z_n}{Z_{n+1}} \leqq x, \end{cases}$$

and $X_n(t) = \sqrt{n}\,(G_n(t) - t)$, then, for each n, the process $X_n(t)$ (the BREIMAN—BRILLINGER representation of the empirical process) has the distribution of the empirical process $Y_n(t)$. The weak convergence of $Y_n(t)$ can be easily proved using the representations $X_n(t)$, while that of $Y_{\nu_n}(t)$ cannot be done the same way. However, the weak convergence of $X_{\nu_n}(t)$ itself is, perhaps, of some interest. In fact, the following theorem is true.

Theorem 2. *If $X_n(t)$ is as above and ν_n, ν, W° are as in Theorem 1, then*

$$X_{\nu_n} \xrightarrow{\mathscr{D}} W^\circ.$$

For the proof of our theorems we will need the following results.

Lemma 1 (Theorem 3 of GUIAŞU [13]). *Suppose that the sequence ν_n and ν are the same as in Theorem 1, and further suppose that the sequence ξ_n of random variables satisfies the following two conditions:*

(i) *For every event A in the σ-algebra, $\mathscr{K}_\nu$, generated by ν, $(P\{A\} > 0)$,*

(8) $$\lim_{n\to\infty} P\{\xi_n \leqq a_n x \mid A\} = F(x),$$

at every continuity point x of the distribution function F. Here a_n is some sequence of positive constants.

(ii) *For every positive ε and η and every A in $\mathscr{K}_\nu (P\{A\} > 0)$, there exist a positive real number $c = c(\varepsilon, \eta)$ and a natural number $n_0 = n_0(\varepsilon, \eta, A)$ such that for every $n \geqq n_0$*

$$P\{\max_{n(1-c) \leqq m \leqq n(1+c)} |\xi_m - \xi_n| > a_n \varepsilon \mid A\} < \eta.$$

Then

$$\lim_{n\to\infty} P\left\{\frac{\xi_{\nu_n}}{a_{\nu_n}} \leqq x\right\} = F(x)$$

at every continuity point x of F.

We remark that Lemma 1, in the above form, differs slightly from the original Theorem 3 of GUIAŞU. The difference is that in his theorem $a_n=1$ for each n. The presence and the use of the sequence a_n in the sense of the above Lemma 1, i.e. a_n not being absorbed into the sequence ξ_n, is needed for the present application in order to make it easier to check for the fulfillment of condition (C6) of GUIAŞU [13], which is replaced here by condition (ii), the original condition of ANSCOMBE [2] if we take $A=\Omega$. For an easy proof adapt the technique of BARNDORFF—NIELSEN [3] to complete the proof of GUIAŞU's Theorem 3 thus modified.

Lemma 2 (Lemma 3 of BLUM, HANSON and ROSENBLATT [5]). *Let η_n be a sequence of independent random variables, further let k_n and m_n, $k_n \leqq m_n$, be two (not constant) sequences of natural numbers. If for each n, A_n is an event depending only on the random variables $\eta_{k_n}, \ldots, \eta_{m_n}$ then for every event A, having positive probability:*

$$\limsup_n P\{A_n|A\} = \limsup_n P\{A_n\}.$$

At a crucial stage, a recent and very important result of J. KIEFER [14], is going to be used in the proof of Theorem 1. His result concerns the representation of the sample distribution function by a SKOROHOD-type embedding in the appropriate two dimensional Gaussian process. Let $\xi(.,.)$ be a Gaussian process on $[0, 1]\times[0, \infty)$ with continuous sample functions, zero expectation, and covariance function

$$E(\xi(s_1, t_1)\xi(s_2, t_2)) = \min(t_1, t_2)[\min(s_1, s_2)-s_1s_2],$$

so that there are independent increments in t and a Brownian bridge in s for fixed t.

Theorem A (J. KIEFER [14]). *ξ can be defined on a probability space on which there is a defined a random function T: $[0, 1]\times[0, \infty)$ such that $\xi(s, T(s, n))$ has the same joint distribution as $\sqrt{n}\,Y_n(s)$ and, as $n\to\infty$,*

$$\frac{1}{\sqrt{n}} \sup_{0\leqq s\leqq 1} \left|\xi(s, T(s, n))-\xi(s, n)\right| = O(n^{-1/6}(\log n)^{2/3})$$

with probability 1.

From now on we assume that the probability space $\{\Omega, \mathscr{B}, P\}$ of the Introduction is already that of Theorem A.

Proof of Theorem 1. To verify that Y_{ν_n} converges weakly to W°, we have to show two things (see Theorem 15.1 in [4] or Theorem A in [10] in these *Acta*): 1) The finite dimensional distributions of Y_{ν_n} converge to those of W°, and 2) The sequence Y_{ν_n} is tight.

Ad 1) As a consequence of relation (1) and the Cramér—Wold device (p. 49 in [4]), if we take the time points $t_1, \ldots, t_k$ and the real numbers $c_1, \ldots, c_k$

($k=1, 2, \ldots$, fixed), then

$$\tilde{R}_n = \sum_{i=1}^{k} c_i Y_n(t_i) \xrightarrow{\mathcal{D}} \sum_{i=1}^{k} c_i W^\circ(t_i) = R.$$

Naturally, $\xrightarrow{\mathcal{D}}$ here stands for convergence in distribution on the real line. By the Cramér—Wold device again, it is enough to show that

$$\tilde{R}_{\nu_n} \xrightarrow{\mathcal{D}} R. \tag{9}$$

Let us introduce the function

$$\psi(x) = \begin{cases} 1, & \text{if } x \geqq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly

$$Y_n(t_i) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (\psi(t_i - U_j) - t_i).$$

The random variables $\psi_{ji} = \psi(t_i - U_j) - t_i$, $i=1, \ldots, k$; $j=1, 2, \ldots$ are independent, $E(\psi_{ji})=0$, $E^2(\psi_{ji})=t_i(1-t_i)$. To verify relation (9), we show now, that the sequence

$$\tilde{R}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} c_i \sum_{j=1}^{n} \psi_{ji} = \frac{R_n}{\sqrt{n}}$$

satisfies the conditions of Lemma 1. As to condition (i) we first remember a well known result of Rényi [18], which states that the necessary and sufficient condition for relation (8) to hold (in the case of $R_n/\sqrt{n}$ instead fo ξ_n/a_n), not only for A's in $\mathcal{K}_\nu$ but for all A in $\mathcal{B}$ (that is, that the sequence $R_n/\sqrt{n}$ should be mixing), is that it should hold for each A of the form

$$A_r = \{R_r \leqq \sqrt{r}\, x\}, \quad r = 1, 2, \ldots .$$

To show this, put

$$\frac{1}{\sqrt{n}} R_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} c_i \sum_{j=p_n}^{n} \psi_{ji},$$

where p_n is a sequence of natural numbers tending to infinity, but so slowly that $p_n/n \to 0$ (e.g. $p_n = [\log n]$). It is obvious, via the Chebishev inequality, that

$$\frac{1}{\sqrt{n}} (R_n - R_n^*) \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty.$$

Thus it is enough to show that, as $n \to \infty$,

$$P\{R_n^* \leqq \sqrt{n}\, x \mid A_r\} \to P\{R \leqq x\}, \tag{10}$$

where r is fixed and not less than an integer n_0, for which it is true that if $n \geqq n_0$, then $P\{A_n\}>0$. But now relation (10) holds, because, if n is so large that $p_n > r \geqq n_0$, then the random variables R_n^* and R_r are independent and thus the conditional probability becomes unconditional.

Turning to the verification of condition (ii) of Lemma 1 we fix the positive ε and η arbitrarily. Clearly

$$(11) \qquad P\{\max_{n(1-c)\leqq m\leqq n(1+c)} |R_m - R_n| > \sqrt{n}\,\varepsilon \,|\, A\} \leqq$$

$$\leqq \sum_{i=1}^{k} P\left\{\max_{n(1-c)\leqq m\leqq n}\left|\sum_{j=1}^{m}\psi_{ji} - \sum_{j=1}^{n}\psi_{ji}\right| > \sqrt{n}\,\frac{\varepsilon}{2k|c_i|}\,\middle|\, A\right\} +$$

$$+ \sum_{i=1}^{k} P\left\{\max_{n\leqq m\leqq n(1+c)}\left|\sum_{j=1}^{m}\psi_{ji} - \sum_{j=1}^{n}\psi_{ji}\right| > \sqrt{n}\,\frac{\varepsilon}{2k|c_i|}\,\middle|\, A\right\}.$$

Now Lemma 2 ensures the existence of a natural number n_1 (which may depend on A) so that if $n \geqq n_1$ then the right-hand side of inequality (11) is not larger than (putting $\theta_i = \frac{\varepsilon}{2k|c_i|}$)

$$(12) \qquad \frac{\eta}{2} + \sum_{i=1}^{k} P\left\{\max_{n(1-c)\leqq m\leqq n}\left|\sum_{j=m+1}^{n}\psi_{ji}\right| > \sqrt{n}\,\theta_i\right\} +$$

$$+ \sum_{i=1}^{k} P\left\{\max_{n\leqq m\leqq n(1+c)}\left|\sum_{j=n+1}^{m}\psi_{ji}\right| > \sqrt{n}\,\theta_i\right\}.$$

Using the KOLMOGOROV inequality, we can now chosoe an integer n_0 $(n_0 \geqq n_1)$ and a real number c (which c does not depend on A) so that for this c and $n \geqq n_0$ the value of formula (12) is less than η. Thus by Lemma 1, the finite dimensional distributions of $Y_{\nu_n}(t)$ converge to those of W°.

Ad 2) As $Y_{\nu_n}(0)=0$, for the tightness of the sequence Y_{ν_n} it is enough to prove (cf. Theorem 15.5 of BILLINGSLEY [4] or Chapter 9, § 8 of GIHMAN and SKOROHOD [12]) that for each positive ε we have

$$(13) \qquad \lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P\{\sup_{|s-t|<\delta} |Y_{\nu_n}(s) - Y_{\nu_n}(t)| > \varepsilon\} = 0.$$

Let θ and ϱ be arbitrary positive numbers and choose a and b, $0<a<b$, so that $P\{a<\nu\leqq b\}>1-\theta$, $\varrho<a$. Since

$$\lim_{n\to\infty} P\left\{\left|\frac{\nu_n}{n} - \nu\right| > \varrho\right\} = 0,$$

the left hand side of (13) is bounded above by

$$\theta + \lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)}\ \sup_{|s-t|<\delta} |Y_m(s) - Y_m(t)| > \varepsilon\}.$$

According to Theorem A we can replace here $Y_m(\cdot)$ by $1/\sqrt{m}\,\xi(\cdot, m)$ on neglectins terms of order $m^{-1/6}(\log m)^{2/3}$ with probability 1. The process $\xi(s, n)$, in turn, ig equivalent in distribution to $X(s, n) - sX(1, n)$, where $X(s, t)$ is a continuous Gaussian process on $[0, 1]\times[0, \infty)$ with zero expectations and independent increments in both directions:

$$E\big(X(s_1, t_1)X(s_2, t_2)\big) = \min(s_1, s_2)\min(t_1, t_2).$$

With this replacement our last expression is less than or equal to

$$\lim_{\delta\to 0}\varlimsup_{n\to\infty} P\left\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)} \frac{1}{\sqrt{m}} \sup_{|s-t|<\delta} |sX(1, m) - tX(1, m)| > \frac{\varepsilon}{2}\right\} +$$

$$+\lim_{\delta\to 0}\varlimsup_{n\to\infty} \sum_{m=[n(a+\varrho)]}^{[n(b+\varrho)]} P\left\{\sup_{|s-t|<\delta} \frac{1}{\sqrt{m}} |X(s, m) - X(t, m)| > \frac{\varepsilon}{2}\right\} + \theta.$$

The first term here is bounded above by

$$\lim_{\delta\to 0}\varlimsup_{n\to\infty} P\left\{\max_{1\leqq m\leqq n(b+\varrho)} \delta|X(1, m)| > \frac{\varepsilon}{2}\sqrt{n(a+\varrho)}\right\} \leqq$$

$$\leqq \lim_{\delta\to 0}\varlimsup_{n\to\infty} \frac{[n(b+\varrho)]\delta^2}{n(a-\varrho)\varepsilon^2} = 0,$$

with Kolmogorov's inequality, while the second one by

$$\text{(14)}\quad \lim_{\delta\to 0}\varlimsup_{n\to\infty} \sum_{m=[n(a-\varrho)]}^{[n(b+\varrho)]} \sum_{i=1}^{\left[\frac{1}{\delta}\right]+1} P\left\{\sup_{(i-1)\delta<s<i\delta} \big|X(s, m) - X((i-1)\delta, m)\big| > \frac{\varepsilon}{6}\sqrt{m}\right\}.$$

Using Theorem 2.1 of Doob [11, p. 392], the probability in (14) is equal to

$$2P\left\{\big|X(i\delta, m) - X((i-1)\delta, m)\big| > \frac{\varepsilon}{6}\sqrt{m}\right\} =$$

$$= 4\int_{\varepsilon\sqrt{m}}^{\infty} \frac{1}{\sqrt{2\pi\delta}} e^{-\frac{x^2}{2\delta}}\,dx \leqq \frac{4\sqrt{\delta}}{\varepsilon\sqrt{m}} \int_{\frac{\varepsilon\sqrt{m}}{\sqrt{\delta}}}^{\infty} \frac{1}{\sqrt{2\pi}} y e^{-\frac{y^2}{2}}\,dy = \frac{\sqrt{\delta}}{\varepsilon\sqrt{m}}\sqrt{\frac{8}{\pi}}\, e^{-\frac{\varepsilon^2 m}{2\delta}};$$

hence (14) is not larger than

$$\lim_{\delta\to 0}\sqrt{\frac{8}{\pi}} \sum_{m=1}^{\infty}\left(\frac{1}{\delta}+1\right)\frac{\sqrt{\delta}}{\varepsilon\sqrt{m}} e^{-\frac{\varepsilon^2 m}{2\delta}} = \sqrt{\frac{8}{\pi\varepsilon^2}}\int_0^{\infty} \lim_{\delta\to 0}\left(\frac{1}{\sqrt{\delta}}+\sqrt{\delta}\right)\frac{1}{\sqrt{x}} e^{-\frac{\varepsilon^2 x}{2\delta}}\,dx = 0.$$

As θ is arbitrary small, Theorem 1 is proved.

Proof of Theorem 2. Let us define the inverse process $X_{\nu_n}^{-1}$ of the process X_{ν_n} of Theorem 2:

$$X_{\nu_n}^{-1}(t) = \sqrt{\nu_n}\left(\frac{Z_i}{Z_{\nu_n+1}} - \frac{i}{\nu_n}\right) \quad \text{for} \quad 1 \leqq i \leqq \nu_n, \quad i-1 < \nu_n t \leqq i.$$

It follows from the definition of the process X_{ν_n} that $X_{\nu_n}^{-1} = -X_{\nu_n}(G_{\nu_n}^{-1})$, where $G_n^{-1}(t) = \inf\{x: G_n(x) \geqq t\}$ which is left continuous, is zero at zero and equals Z_i/Z_{n+1} at i/n. Now

$$X_{\nu_n}^{-1}(t) = \frac{\nu_n}{Z_{\nu_n+1}}\left(\frac{Z_i - i}{\sqrt{\nu_n}} - \frac{i}{\nu_n}\frac{Z_{\nu_n+1} - \nu_n}{\sqrt{\nu_n}}\right) \quad \text{for} \quad 1 \leqq i \leqq \nu_n, \quad i-1 < \nu_n t \leqq i.$$

Because $E(\xi_1) = 1$, $E((\xi_1 - 1)^2) = 1$, it follows that

$$\frac{\nu_n}{Z_{\nu_n+1}} \xrightarrow{P} 1$$

and

$$\frac{\xi_{\nu_n+1}}{\sqrt{\nu_n}} \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty.$$

Consequently if we define $S_i = Z_i - i$ for $i = 1, 2, \dots$; $S_0 = 0$, and $S^{(n)}(t) = S_{[nt]}/\sqrt{n}$ for $0 \leqq t \leqq 1$ and $n = 1, 2, \dots$, then the distribution of $X_{\nu_n}^{-1}(t)$ for large n is equal to that of $S^{(\nu_n)}(t) - tS^{(\nu_n)}(1)$, that is, for large n, $X_{\nu_n}^{-1}$ is a partial sum type process of [10]. And, as X_n^{-1} is known to converge weakly to a Brownian bridge W_1° also

$$X_{\nu_n}^{-1} \xrightarrow{\mathcal{D}} W_1^\circ,$$

as a consequence of Theorem 1 of [10]. Because

$$\sup_{0 \leqq t \leqq 1} \left(G_{\nu_n}^{-1}(t) - t\right) \xrightarrow{P} 0,$$

it follows that

$$X_{\nu_n} \xrightarrow{\mathcal{D}} -W_1^\circ = W^\circ;$$

since the negative of a Brownian bridge is again a Brownian bridge, this also completes the proof of Theorem 2.

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